

LAGRANGIAN BONNET PAIRS IN COMPLEX SPACE FORMS

HUIXIA HE, HUI MA, AND ERXIAO WANG

ABSTRACT. In this paper we first give a Bonnet theorem for conformal Lagrangian surfaces in complex space forms, then we show that any compact Lagrangian surface in the complex space form admits at most one other global isometric Lagrangian surface with the same mean curvature form, unless the Maslov form is conformal. These two Lagrangian surfaces are then called Lagrangian Bonnet pairs. We also studied the question about Lagrangian Bonnet surfaces in $\tilde{M}^2(4c)$, and obtain some interesting results.

1. INTRODUCTION

In surface theory of 3-dimensional Euclidean space \mathbb{R}^3 , the series work by Bonnet, Cartan and Chern ([2, 7, 8]) show that mean curvature and metric is sufficient to determine an oriented surface generically, except three cases: *constant mean curvature surfaces*, *Bonnet surfaces*, which admits a nontrivial isometric deformation preserving the mean curvature function, and *Bonnet pairs*, which are exactly two non-congruent isometric surfaces with the same mean curvature function. CMC surfaces have been investigated intensively by various methods; Bonnet surfaces have been treated using techniques of integrable systems theory and extended to S^3 and H^3 , and recently have been generalized to the homogeneous 3-manifold with a 4-dimensional isometry group have been investigated intensively by various methods ([7], [8], [9], [10], [14]). But much less is known about Bonnet pairs. Bobenko [1] takes some steps towards attacking this problem by treating Bonnet pairs as integrable systems. However, it is still an open question whether compact Bonnet pairs exist in \mathbb{R}^3 . Since the theory of Bonnet pairs in \mathbb{R}^3 is closely related to isothermic surfaces in S^3 , it was generalized to S^3 in [17]. Lawson and Tribuzy ([18]) showed that any compact oriented surface in the 3-dimensional real form $M^3(c)$ with nonconstant mean curvature, admits at most two surfaces with the given metric and mean curvature. Springborn ([24]) showed that helicoidal immersed tori are compact Bonnet pairs in 3-dimensional sphere.

By the investigation of Lagrangian surfaces in the complex projective plane \mathbb{CP}^2 , we find it is very interesting to consider the analog problem for conformal Lagrangian surfaces in the complex space form $\tilde{M}^2(4c)$, where $c = 1, 0$ or -1 . Explicitly, which data are sufficient to determine a conformal Lagrangian immersion of a Riemannian surface in $\tilde{M}^2(4c)$ up to rigidity motions. The first two authors introduced a new concept of *Lagrangian Bonnet pairs* in \mathbb{CP}^2 in a similar spirit and derived a Lawson-Tribuzy type theorem ([15]). In this paper, we first generalize this result to the surfaces in the complex Euclidean plane \mathbb{C}^2 and the complex hyperbolic plane \mathbb{CH}^2 . On the other hand, we also study the *Lagrangian Bonnet surfaces* in $\tilde{M}^2(4c)$, i.e. to Lagrangian surfaces in complex space forms possessing one-parameter families of isometries preserving the mean curvature form, and obtain several interesting results.

Remark that Lagrangian submanifolds in complex space forms with conformal Maslov form were deeply studied in [3], [4], [5], [6], [21] and etc.

2010 *Mathematics Subject Classification*. Primary 53C40, Secondary 53C42, 53D12.

Key words and phrases. Lagrangian submanifolds, complex space forms, Bonnet pairs.

*Corresponding author: Huixia He.

2. LAGRANGIAN SURFACES IN COMPLEX SPACE FORMS

2.1. Lagrangian surface in \mathbb{C}^2 . Identify \mathbb{C}^2 with \mathbb{R}^4 equipped with its standard inner product $\langle \cdot, \cdot \rangle$, the standard complex structure J and Kähler form ω . Let $f : \Sigma \rightarrow \mathbb{C}^2$ be a Lagrangian immersion of an oriented surface. The induced metric on Σ generates a complex structure with respect to which the metric is $g = 2e^u dz d\bar{z}$, where $z = x + iy$ is a local complex coordinate on Σ and u is a real-valued function defined on Σ locally. Let f_z and $f_{\bar{z}}$ denote the complexified tangent vector, where

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Complexify the product $\langle \cdot, \cdot \rangle$ and the complex structure on \mathbb{R}^4 to \mathbb{C}^4 , still denoted by $\langle \cdot, \cdot \rangle$ and J . Then the metric g is conformal that gives

$$(2.1) \quad \langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0, \quad \langle f_z, f_{\bar{z}} \rangle = e^u.$$

Moreover, a conformal immersion $f : \Sigma \rightarrow \mathbb{C}^2$ is Lagrangian if and only if

$$(2.2) \quad \langle f_z, Jf_{\bar{z}} \rangle = 0.$$

Thus the vectors $f_z, f_{\bar{z}}, Jf_z$ as well as $Jf_{\bar{z}}$ define an orthogonal moving frame on the surface. Denote $\sigma = (f_z, f_{\bar{z}}, Jf_z, Jf_{\bar{z}})$, which due to (2.1) and (2.2) satisfies the following equations

$$(2.3) \quad \sigma_z = \sigma \mathcal{U}, \quad \sigma_{\bar{z}} = \sigma \mathcal{V},$$

$$(2.4) \quad \mathcal{U} = \begin{pmatrix} u_z & 0 & -\phi & -\bar{\phi} \\ 0 & 0 & -e^{-u}\psi & -\phi \\ \phi & \bar{\phi} & u_z & 0 \\ e^{-u}\psi & \phi & 0 & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & -\bar{\phi} & -e^{-u}\bar{\psi} \\ 0 & u_{\bar{z}} & -\phi & -\bar{\phi} \\ \bar{\phi} & e^{-u}\bar{\psi} & 0 & 0 \\ \phi & \bar{\phi} & 0 & u_{\bar{z}} \end{pmatrix},$$

where

$$\phi = e^{-u} \langle f_{z\bar{z}}, Jf_z \rangle, \quad \psi = \langle f_{zz}, Jf_z \rangle.$$

The one-form $\Phi = \phi dz$ and the cubic differential $\Psi = \psi dz^3$ are globally defined on Σ which are independent of the choice of the complex coordinates and $U(2)$ -invariant. We call Φ and Ψ mean curvature form and Hopf differential of f , respectively.

By straightforward calculations we get the following integrability conditions:

$$\begin{aligned} \phi_{\bar{z}} - \bar{\phi}_z &= 0, \\ u_{z\bar{z}} + |\phi|^2 - e^{-2u}|\psi|^2 &= 0, \\ e^{-2u}\psi_{\bar{z}} &= (e^{-u}\phi)_z. \end{aligned}$$

2.2. Lagrangian surface in $\mathbb{C}H^2$. Let $(\mathbb{C}H^2, h, J, \omega)$ be the 2-dimensional complex hyperbolic space endowed with the Fubini-Study metric h of constant holomorphic sectional curvature -4 , where J denotes the standard complex structure of $\mathbb{C}H^2$ and ω is the Kähler form given by $\omega(X, Y) = h(JX, Y)$ for any tangent vectors X and Y .

Let

$$\mathbb{H}_1^5 = \{Z \in \mathbb{C}^3 \mid (Z, Z) = -1\},$$

where (\cdot, \cdot) denotes the Hermitian inner product defined by

$$(Z, W) = -z_0\bar{w}_0 + z_1\bar{w}_1 + z_2\bar{w}_2,$$

for any $Z = (z_0, z_1, z_2), W = (w_0, w_1, w_2) \in \mathbb{C}^3$. Thus $\langle Z, W \rangle = \operatorname{Re}(z, w)$ induces a metric on \mathbb{H}_1^5 of constant curvature -1 . Hence the Fubini-Study metric h on $\mathbb{C}H^2$ is obtained from the fact that the Hopf fibration $\pi : \mathbb{H}_1^5(-1) \rightarrow \mathbb{C}H^2$ is a Riemannian submersion.

Now let $f : \Sigma \rightarrow \mathbb{C}H^2$ be a Lagrangian immersion of an oriented surface. Still denote the induced metric $g = f^*h = 2e^u dz d\bar{z}$ on Σ by $\langle \cdot, \cdot \rangle$ without causing ambiguity. And there always exists a horizontal local lift $F : U \rightarrow \mathbb{H}_1^5$ such that

$$(2.5) \quad \langle F_z, JF \rangle = \langle F_{\bar{z}}, JF \rangle = 0,$$

where U is an open set of Σ . In fact, generally, it follows from Σ is Lagrangian that $\langle dF, JF \rangle$ is a closed one-form for any local lift F . So there exists a real function $\eta \in C^\infty(U)$ locally such that $d\eta = \langle dF, JF \rangle$. Then $\tilde{F} = e^{i\eta} F$ is a horizontal local lift for f to \mathbb{H}_1^5 .

The metric g is conformal that gives

$$(2.6) \quad \langle F_z, F_{\bar{z}} \rangle = e^u,$$

$$(2.7) \quad \langle F_z, F_z \rangle = \langle F_{\bar{z}}, F_{\bar{z}} \rangle = 0.$$

F is a Lagrangian immersion means that

$$\langle F_z, JF_{\bar{z}} \rangle = \langle F_{\bar{z}}, JF_z \rangle = 0.$$

Thus the vectors $F, JF, F_z, JF_z, F_{\bar{z}}$ as well as $JF_{\bar{z}}$ define a moving frame on the surface. One obtains the following frame equations:

$$(2.8) \quad \begin{aligned} F_{zz} &= u_z F_z + \phi JF_z + e^{-u} \psi JF_{\bar{z}}, \\ F_{z\bar{z}} &= \bar{\phi} JF_z + \phi JF_{\bar{z}} + e^u F, \\ F_{\bar{z}\bar{z}} &= u_{\bar{z}} F_{\bar{z}} + e^{-u} \bar{\psi} JF_z + \bar{\phi} JF_{\bar{z}}, \end{aligned}$$

where

$$(2.9) \quad \phi = e^{-u} \langle F_{z\bar{z}}, JF_z \rangle, \quad \psi = \langle F_{zz}, JF_z \rangle.$$

It is easy to see that the one-form $\Phi = \phi dz$ and the cubic differential $\Psi = \psi dz^3$ are globally defined on Σ and $U(1, 2)$ -invariant. We also call Φ and Ψ mean curvature form and Hopf differential of f , respectively.

The compatibility condition of equations (2.8) has the following form:

$$\begin{aligned} \phi_{\bar{z}} - \bar{\phi}_z &= 0, \\ u_{z\bar{z}} + |\phi|^2 - e^u - e^{-2u} |\psi|^2 &= 0, \\ e^{-2u} \psi_{\bar{z}} &= (e^{-u} \phi)_z. \end{aligned}$$

Recall that for a conformal Lagrangian immersed surface $f : \Sigma \rightarrow \mathbb{C}P^2(4)$, let F be the horizontal local lift to $S^5(1)$ ([15]). Still denote by $\langle \cdot, \cdot \rangle$ the standard inner product on \mathbb{C}^3 and J the standard complex structure. Define the mean curvature form $\Phi = \phi dz$ and the cubic Hopf cubic form $\Psi = \psi dz^3$ by (2.9) in terms of the corresponding F and $\langle \cdot, \cdot \rangle$. Then we have

Theorem 2.1. *Let $f : \Sigma \rightarrow \tilde{M}^2(4c)$ be a conformal Lagrangian surface in $\tilde{M}^2(4c)$ with the metric $g = 2e^u dz d\bar{z}$ for $c = 0, 1$ or -1 . Then the metric g , mean curvature form $\Phi = \phi dz$ and cubic Hopf differential $\Psi = \psi dz^3$ satisfy the following equations:*

$$(2.10) \quad \phi_{\bar{z}} - \bar{\phi}_z = 0,$$

$$(2.11) \quad u_{z\bar{z}} + |\phi|^2 + c e^u - e^{-2u} |\psi|^2 = 0,$$

$$(2.12) \quad e^{-2u} \psi_{\bar{z}} = (e^{-u} \phi)_z.$$

Conversely, given a metric $g = 2e^u dz d\bar{z}$, a one-form $\Phi = \phi dz$ and a cubic form $\Psi = \psi dz^3$ on Σ satisfying (2.10), (2.11) and (2.12), there exists a Lagrangian immersion $f : \tilde{\Sigma} \rightarrow \tilde{M}^2(4c)$ from the universal cover $\tilde{\Sigma}$ of Σ to $\tilde{M}^2(4c)$ with the metric g and mean curvature form Φ and cubic Hopf differential Ψ . The immersion f is unique up to isometries in $\tilde{M}^2(4c)$.

- Remark 1.** (1) Instead of the 1-form $\Phi = \phi dz$, Oh [22], Schoen and Wolfson [23] used the famous Maslov form σ_H defined by $\sigma_H = \omega(H, \cdot)$ and in fact $\sigma_H = -(\Phi + \bar{\Phi})$, where H is the mean curvature vector field, or Castro and Urbano introduced a global vector field $\mathcal{K} = e^{-u}\bar{\phi}\frac{\partial}{\partial z}$ over Σ . Remark that $|\phi|^2 = \frac{1}{2}|H|^2e^u$.
- (2) It is known ([12]) that the Maslov form of a Lagrangian submanifold in an Einstein-Kähler manifold is always closed. This is equivalent to the first integrability equation (2.10) in our cases.
- (3) Actually the cubic differential Ψ has been introduced by several authors in the study of minimal surfaces in Kähler manifolds, for instance, Eells and Wood [13], Chern and Wolfson [11], etc. It corresponds to the symmetric 3-tensor field $S(X, Y, Z) := h(II(X, Y), JZ)$ for a Lagrangian submanifold in a Kähler manifold, where II is the second fundamental form of the submanifold and h is the Kähler metric tensor of the ambient manifold (refer to, e.g., [4]).
- (4) In order to be consistent with the notations in most publications on Lagrangian surfaces, such as [5, 4], we define Φ and Ψ here in terms of the complexified standard inner products. In fact, this definition is same as the one we used in [20, 15] given by the Hermitian inner product, only up to a factor $\sqrt{-1}$.

Combining Bonnet theorems, we can summarize the geometric interpretations of Φ and Ψ (refer to [4, 5, 6, 15, 16, 19, ?]):

Proposition 2.2. *Let $f : \Sigma \rightarrow \tilde{M}(4c)$ be an oriented immersed surface Σ . Then f is minimal if and only if $\Phi \equiv 0$. In addition, if f is Lagrangian, then the following statements are equivalent:*

- (1) f is Hamiltonian stationary.
- (2) The Maslov form σ_H is a harmonic 1-form.
- (3) The Lagrangian angle β is a local harmonic function.
- (4) The mean curvature form Φ is holomorphic.

Proposition 2.3. *Let $f : \Sigma^2 \rightarrow \tilde{M}^2(4c)$ be a conformal Lagrangian immersion with the induced metric $g = 2e^u dz d\bar{z}$. Then the following statements are equivalent:*

- (1) The Maslov form σ_H is conformal.
- (2) The vector field JH is a conformal vector field.
- (3) The cubic Hopf differential Ψ is holomorphic.
- (4) $(e^{-u}\phi)_z = 0$.

3. LAGRANGIAN BONNET SURFACES IN COMPLEX SPACE FORMS

We now suppose that we are given three isometric noncongruent Lagrangian immersions $f_k : \Sigma \rightarrow \tilde{M}(4c)$, $k = 1, 2, 3$ with coinciding mean curvature one-form Φ . As conformal immersions of the same Riemann surface, they are described by the corresponding cubic Hopf differentials Ψ_1, Ψ_2, Ψ_3 , the conformal metric $2e^u dz d\bar{z}$ and the mean curvature form Φ . Since the surfaces are non-congruent the cubic Hopf differentials differ.

It follows from (2.11) and (2.12) that

Proposition 3.1. *Each of the differences $\Psi_{ij} \equiv \Psi_i - \Psi_j$ for $1 \leq i, j \leq 3$, is a holomorphic cubic differential form on Σ . Moreover,*

$$(3.1) \quad |\Psi_i| = |\Psi_j|, \quad 1 \leq i, j \leq 3.$$

Due to the second statement of Proposition 3.1 the zeros of Ψ_k for $k = 1, 2, 3$ coincide. Clearly, these points are contained in the zeros of the holomorphic differential Ψ_{ij} . We call this isolated

points set as the umbilic points set of $f_k, k = 1, 2, 3$. Denote it by

$$\mathcal{U} = \{P \in \Sigma \mid \Psi_k(P) = 0\}.$$

Let Σ be a Lagrangian surface in \tilde{M} with the data $\{u, \Phi, \Psi\}$ and Σ^* be an isometric deformation of Σ preserving the mean curvature form Φ with the data $\{u, \Phi, \Psi^*\}$.

The following result is the Lagrangian version of Tribuzy's result (see [9] or [15], [25]).

Lemma 3.2. *Let Σ be a Lagrangian Bonnet surface in $\tilde{M}(c)$, then*

$$(3.2) \quad (\log \psi)_{z\bar{z}} = |(\log \psi)_{\bar{z}}|^2.$$

and it is equivalent to

$$(3.3) \quad \left(\frac{\psi_{\bar{z}}}{|\psi|^2}\right)_z = 0,$$

Proof. From (2.11), we have $\psi^* = e^{it}\psi$, where t is a real-valued function determined up to a multiple of 2π . Then (2.12) implies that

$$(e^{it}\psi - \psi)_{\bar{z}} = 0,$$

which is equivalent to

$$(3.4) \quad (e^{it} - 1)\psi_{\bar{z}} + e^{it}it_{\bar{z}}\psi = 0.$$

Thus

$$(3.5) \quad \tau \equiv dt - (1 - e^{-it})i\frac{\psi_{\bar{z}}}{\psi}d\bar{z} + (1 - e^{it})i\frac{\bar{\psi}_z}{\bar{\psi}}dz = 0.$$

The Pfaff system $\tau = 0$ is completely integrable if and only if $d\tau \wedge \tau = 0$, that is,

$$e^{it} \left[\left| \frac{\psi_{\bar{z}}}{\psi} \right|^2 - \left(\frac{\bar{\psi}_z}{\bar{\psi}} \right)_{\bar{z}} \right] + e^{-it} \left[\left| \frac{\bar{\psi}_z}{\bar{\psi}} \right|^2 - \left(\frac{\psi_{\bar{z}}}{\psi} \right)_z \right] + \left(\frac{\psi_{\bar{z}}}{\psi} \right)_z + \left(\frac{\bar{\psi}_z}{\bar{\psi}} \right)_{\bar{z}} - \left| \frac{\psi_{\bar{z}}}{\psi} \right|^2 - \left| \frac{\bar{\psi}_z}{\bar{\psi}} \right|^2 = 0.$$

Then (3.3) follows from the arbitrariness of t . □

Corollary 3.3. *Let Σ be a Lagrangian Bonnet surface in $\tilde{M}(c)$, then*

$$(3.6) \quad \left(\frac{(e^{-u}\phi)_z}{|\phi|^2 + e^u(c - K)} \right)_z = 0.$$

Writing $\psi = |\psi|e^{i\alpha}$. Then from (3.2), (2.11) and (2.12) we get

Corollary 3.4. *Let Σ be a Lagrangian Bonnet surface in $\tilde{M}(c)$, then*

$$(3.7) \quad (\log(e^{3u}(e^{-u}|\phi|^2 + c - K)))_{z\bar{z}} - \frac{2e^u|(e^{-u}\phi)_z|^2}{e^{-u}|\phi|^2 + c - K} = 0,$$

$$(3.8) \quad \alpha_{z\bar{z}} = 0.$$

Definition 3.5. A Lagrangian surface Σ in \tilde{M} is isothermic if and only if there exists locally a conformal parameter z such that ψdz^3 satisfies $\psi(z, \bar{z}) \in \mathbb{R}$.

Proposition 3.6. *A Lagrangian surface Σ in \tilde{M} is isothermic if and only if $\text{Im}(\log \psi)_{z\bar{z}} = 0$.*

Proposition 3.7. *A Lagrangian surface Σ in \tilde{M} is a Lagrangian Bonnet surface if and only if*

- (1) Σ is Lagrangian isothermic,
- (2) (3.7) holds.

Theorem 3.8. *A Lagrangian surface Σ in \tilde{M} is a Lagrangian Bonnet surface. Then*

- (1) Σ is Lagrangian isothermic,

(2) $\frac{1}{\psi}$ is harmonic with respect to its isothermic coordinate, i.e. $(\frac{1}{\psi})_{z\bar{z}} = 0$.

Let Σ be a Lagrangian Bonnet surface in \tilde{M} . We can choose isothermic coordinate z such that $1/\psi$ is a real-valued harmonic function of z . From $(\frac{1}{\psi})_{z\bar{z}} = 0$, we have

$$(3.9) \quad \frac{1}{\psi} = h + \bar{h}$$

for some holomorphic function h .

Substituting (3.9) into (2.12), we get

$$h_z(e^{-u}\phi)_z = \bar{h}_{\bar{z}}(e^{-u}\bar{\phi})_{\bar{z}}.$$

The Codazzi equation (2.12) implies

$$(3.10) \quad e^{2u} = \frac{\psi_{\bar{z}}}{(e^{-u}\phi)_z} = -\frac{\bar{h}_{\bar{z}}}{(h + \bar{h})^2(e^{-u}\phi)_z} = -\frac{h_z}{(h + \bar{h})^2(e^{-u}\bar{\phi})_{\bar{z}}}.$$

Differentiating it twice, we have

$$(3.11) \quad 2u_z = \frac{h_{zz}}{h_z} - \frac{2h_z}{h + \bar{h}} - \frac{(e^{-u}\bar{\phi})_{\bar{z}z}}{(e^{-u}\bar{\phi})_{\bar{z}}}$$

$$(3.12) \quad 2u_{z\bar{z}} = \frac{h_{z\bar{z}}}{h_z} - \frac{h_{zz}h_{z\bar{z}}}{(h_z)^2} - \frac{2h_{z\bar{z}}}{h + \bar{h}} + \frac{2h_z\bar{h}_{\bar{z}}}{(h + \bar{h})^2} - \left(\frac{(e^{-u}\phi)_{\bar{z}z}}{(e^{-u}\bar{\phi})_{\bar{z}}} \right)_{\bar{z}}$$

$$(3.13) \quad = \frac{2|h_z|^2}{(h + \bar{h})^2} - \left(\frac{(e^{-u}\phi)_{\bar{z}z}}{(e^{-u}\bar{\phi})_{\bar{z}}} \right)_{\bar{z}}$$

From Gauss equation we know that

$$\begin{aligned} u_{z\bar{z}} &= e^{-2u}|\psi|^2 - (|\phi|^2 + ce^u) = e^{-2u}|\psi|^2 - e^{2u}(|e^{-u}\phi|^2) - ce^u \\ &= -\frac{(e^{-u}\bar{\phi})_{\bar{z}}}{h_z} + \frac{h_{\bar{z}}}{(h + \bar{h})^2} \frac{|e^{-u}\phi|^2}{(e^{-u}\bar{\phi})_{\bar{z}}} - ce^u \end{aligned}$$

Substituting (3.10) and (3.12) into Gauss equation, we get

$$(3.14) \quad \left(\frac{(e^{-u}\phi)_{\bar{z}z}}{(e^{-u}\bar{\phi})_{\bar{z}}} \right)_{\bar{z}} - \frac{(e^{-u}\bar{\phi})_{\bar{z}}}{h_z} = \frac{2|h_z|^2}{(h + \bar{h})^2} - \frac{2h_z|e^{-u}\phi|^2}{(h + \bar{h})^2(e^{-u}\bar{\phi})_{\bar{z}}} + 2ce^u$$

If ϕ is a real-value function on Σ , we have the following result.

Theorem 3.9. *Let Σ be a Lagrangian Bonnet surface in \mathbb{C}^2 with isothermic coordinates z, \bar{z} . Then*

$$w = w(z) = \int \frac{1}{h_z(z)} dz$$

is also a conformal coordinate, and the mean curvature form $e^{-u}\phi$, $\frac{e^{2u}}{h_z}$ and $\frac{|\psi|}{|h_z|}$ are functions of

$$t = w + \bar{w}$$

only.

Proof 1. By the chain rule we get that $(e^{-u}\phi)_w = h_z(e^{-u}\phi)_z$ and $(e^{-u}\phi)_{\bar{w}} = \bar{h}_{\bar{z}}(e^{-u}\phi)_{\bar{z}}$ which implies that

$$(e^{-u}\phi)_w = (e^{-u}\bar{\phi})_{\bar{w}}.$$

Put $2w = t + is$, then we get

$$(e^{-u}\phi)_t = (e^{-u}\phi)_w = (e^{-u}\phi)_{\bar{w}}$$

which shows that $(e^{-u}\phi)$ depends on one variable t only.

$$e^{2u} = \frac{\psi_{\bar{z}}}{(e^{-u}\phi)_z} = -\frac{|h_z|^2}{(h + \bar{h})^2(e^{-u}\phi)_w}.$$

$$(e^{-u}\phi)_{z\bar{z}} = (e^{-u}\phi)_w w_z \bar{z} = (e^{-u}\phi)_{w\bar{w}} w_z \bar{w} \bar{z} = \frac{(e^{-u}\phi)_{w\bar{w}}}{|h_z|^2}$$

Therefore, we have

$$\left(\frac{(e^{-u}\phi)_{\bar{z}z}}{(e^{-u}\phi)_{\bar{z}}} \right)_{\bar{z}} = \frac{1}{|h_z|^2} \left(\frac{(e^{-u}\phi)''_t}{(e^{-u}\phi)'_t} \right)'_t$$

Then (3.14) can be rewritten as

$$\begin{aligned} \frac{1}{|h_z|^2} \left[\left(\frac{(e^{-u}\phi)''_t}{(e^{-u}\phi)'_t} \right)'_t - (e^{-u}\phi)'_t \right] &= \frac{2|h_z|^2}{(h+\bar{h})^2} - \frac{2|h_z|^2|e^{-u}\phi|^2}{(h+\bar{h})^2(e^{-u}\phi)'_t} + 2c \frac{|h_z|}{(h+\bar{h})\sqrt{-(e^{-u}\phi)'_t}} \\ \Rightarrow \left(\frac{(e^{-u}\phi)''_t}{(e^{-u}\phi)'_t} \right)'_t - 2(e^{-u}\phi)'_t &= \frac{2|h_z|^4}{(h+\bar{h})^2} \left(1 - \frac{|e^{-u}\phi|^2}{(e^{-u}\phi)'_t} \right) + 2c \frac{|h_z|^3}{(h+\bar{h})\sqrt{-(e^{-u}\phi)'_t}} \end{aligned}$$

When $c = 0$, we can get

$$(3.15) \quad \left(\frac{(e^{-u}\phi)''_t}{(e^{-u}\phi)'_t} \right)'_t - 2(e^{-u}\phi)'_t = \frac{2|h_z|^4}{(h+\bar{h})^2} \left(1 - \frac{(e^{-u}\phi)^2}{(e^{-u}\phi)'_t} \right) = 2Q^2 \left(1 - \frac{(e^{-u}\phi)^2}{(e^{-u}\phi)'_t} \right)$$

where $Q = \frac{|h_z|^2}{(h+\bar{h})}$. This formula implies that Q is a function of t unless $\frac{(e^{-u}\phi)''_t}{(e^{-u}\phi)'_t} - 2(e^{-u}\phi)'_t$ and $1 - \frac{(e^{-u}\phi)^2}{(e^{-u}\phi)'_t}$ must vanish identically. $(e^{-u}\phi)'_t = (e^{-u}\phi)^2$ means that $(e^{-u}\phi)'_t > 0$, but this is a contradiction to (3.10).

Since $\tilde{\psi}(z, \bar{z})dz^3 = \psi(w, \bar{w})dw^3$, where $\tilde{\psi}$ is the Hopf differential form with respect to the coordinates z, \bar{z} , we get

$$\begin{aligned} \psi(w, \bar{w}) &= \frac{h_z^3(w^{-1}(w))}{h(w^{-1}(w)) + \bar{h}(\bar{w}^{-1}(\bar{w}))}, \\ \psi_{\bar{w}} &= \frac{h_z|h_z|^4}{(h+\bar{h})^2}, \bar{\psi}_w = \frac{\bar{h}_{\bar{z}}|h_z|^4}{(h+\bar{h})^2} \\ |\psi|^2 &= \psi_{\bar{w}}\bar{\psi}_w = \bar{\psi}_w h_z = \frac{|h_z|^6}{(h+\bar{h})^2} = |h_z|^2 Q^2. \end{aligned}$$

Then $\frac{2|\psi|^2}{|h_z|^2}$ is a function of t only.

$$\frac{e^{2u}}{h_z} = \frac{\psi_{\bar{w}}}{h_z(e^{-u}\phi)_w} = \frac{Q^2}{(e^{-u}\phi)'_t}$$

It's depends on t only, too.

Theorem 3.10. *The holomorphic function $h = h(z)$ satisfies the differential equation*

$$h_{zz}(h+\bar{h}) - h_z^2 = \bar{h}_{\bar{z}\bar{z}}(h+\bar{h}) - \bar{h}_{\bar{z}}^2$$

Proof 2. Since $(e^{-u}\phi)'_t - (e^{-u}\phi)^2 \neq 0$, and $Q = \frac{|h_z|^2}{(h+\bar{h})}$ depends on t only, i.e.

$$\begin{aligned} Q_w = Q_{\bar{w}} &\Leftrightarrow h_z Q_z = \bar{h}_{\bar{z}} Q_{\bar{z}}. \\ Q_z &= \frac{h_{zz}\bar{h}_{\bar{z}}}{(h+\bar{h})} - |h_z|^2 \frac{h_z}{(h+\bar{h})^2}, \quad h_z Q_z = \frac{h_{zz}|h_z|^2}{(h+\bar{h})} - \frac{|h_z|^2(h_z)^2}{(h+\bar{h})^2} \\ Q_{\bar{z}} &= \frac{h_z\bar{h}_{\bar{z}\bar{z}}}{(h+\bar{h})} - |h_z|^2 \frac{\bar{h}_{\bar{z}}}{(h+\bar{h})^2}, \quad \bar{h}_{\bar{z}} Q_{\bar{z}} = \frac{\bar{h}_{\bar{z}\bar{z}}|h_z|^2}{(h+\bar{h})} - \frac{|h_z|^2(\bar{h}_{\bar{z}})^2}{(h+\bar{h})^2} \end{aligned}$$

This is equivalent to

$$\frac{\bar{h}_{\bar{z}\bar{z}}}{(h+\bar{h})} - \frac{(\bar{h}_{\bar{z}})^2}{(h+\bar{h})^2} = \frac{h_{zz}}{(h+\bar{h})} - \frac{(h_z)^2}{(h+\bar{h})^2}$$

then we finish the proof.

Theorem 3.11. *Let Σ be a compact oriented Lagrangian surface in $\tilde{M}^2(4c)$. If its Maslov form is not conformal, then there exist at most two noncongruent isometric immersions of Σ in $\tilde{M}^2(4c)$ with the mean curvature form Φ .*

Proof 3. From now on we assume that f_1, f_2 and f_3 are mutually noncongruent. We will only use the fact for the two Lagrangian immersions f_1 and f_2 . Considering $|\psi_1|^2 = |\psi_2|^2$, we may write

$$\psi_2 = \psi_1 e^{i\theta},$$

where θ is well defined outside the zeros of $|\psi_k|^2 = e^{3u}(e^{-u}|\phi|^2 + c - K) = e^{3u}(\frac{1}{2}|H|^2 + c - K)$ modulo 2π , where K is the Gauss curvature of the induced metric and $|H|$ is the length of the mean curvature vector.

We now consider

$$Q := \frac{\psi_1 - \psi_2}{\psi_1} = 1 - e^{i\theta},$$

which is well defined on $\Sigma \setminus \mathcal{U}$. It follows from Lemma 3.2 and $\psi_1 - \psi_2$ is holomorphic that

$$(3.16) \quad \Delta \log Q = -\Delta \log \psi_1 \leq 0,$$

where Δ is the Laplacian operator on Σ . From (3.16) and $\Delta \log Q = \Delta \log |Q| + i\Delta \arg Q$, we know that

$$\Delta \log |Q| \leq 0, \quad \Delta \arg Q = 0$$

on $\Sigma \setminus \mathcal{U}$.

We now observe that since Q is not zero in the connected set $\Sigma \setminus \mathcal{U}$, the function θ cannot be zero modulo 2π in this set. Hence we can choose a continuous branch $\theta : \Sigma \setminus \mathcal{U} \rightarrow (0, 2\pi)$. Then there exists a continuous branch

$$\arg(Q(z)) \in (-\frac{\pi}{2}, \frac{\pi}{2}),$$

for $z \in \Sigma \setminus \mathcal{U}$. In particular, $\arg Q$ is a bounded harmonic function on $\Sigma \setminus \mathcal{U}$, where \mathcal{U} is a discrete points set. Therefore, by removable singularities theorem, $\arg Q$ can extend to a smooth harmonic function on Σ . Σ is compact and connected, hence $\arg Q$ is a constant. Moreover, $|Q - 1| \equiv 1$, which follows that Q is a constant. Consequently, Ψ_1 is holomorphic, then by Proposition 2.3, its Maslov form is conformal. This completes the proof.

4. LAGRANGIAN BONNET PAIRS

Let f_1, f_2 be a Lagrangian Bonnet pair, i.e., two isometric noncongruent Lagrangian surfaces with coinciding mean curvature form. As conformal immersions of the same Riemann surface

$$f_1 : \Sigma \rightarrow \tilde{M}(4c), \quad f_2 : \Sigma \rightarrow \tilde{M}(4c),$$

they are described by the corresponding cubic Hopf differentials Ψ_1, Ψ_2 , the conformal metric $2e^u dz d\bar{z}$ and the mean curvature form Φ . Since the surfaces are non-congruent the cubic Hopf differentials differ $\Psi_1 \neq \Psi_2$.

Proposition 4.1. *Let Ψ_1 and Ψ_2 be the cubic Hopf differentials of a Lagrangian Bonnet pair $f_{1,2} : \Sigma \rightarrow \tilde{M}(4c)$. Then there exist a holomorphic cubic differential $h = \Psi_1 - \Psi_2$ on Σ and a smooth real valued function $\alpha : \Sigma \rightarrow \mathbb{R}$ such that*

$$\Psi_1 = \frac{1}{2}h(i\alpha + 1), \quad \Psi_2 = \frac{1}{2}h(i\alpha - 1).$$

Proof. Define a smooth cubic differential

$$q = \Psi_1 + \Psi_2.$$

(3.1) implies

$$q\bar{h} + h\bar{q} = 0.$$

Thus $\alpha = -i\frac{q}{h}$ is a real valued function defined on $\Sigma \setminus \mathcal{U}_h$ where

$$\mathcal{U}_h = \{P \in \Sigma | h(P) = 0\}$$

is the zero set of h . At any $z_0 \in \mathcal{U}_h$ the holomorphic differential h has the form

$$h(z) = (z - z_0)^k h_0(z) dz, \quad h_0(z_0) \neq 0, \quad k \in \mathbb{N}.$$

In a neighborhood of z_0 we have

$$\alpha = -\frac{i}{(z - z_0)^k} \frac{q(z)}{h_0(z)}$$

where q is smooth and h_0 is holomorphic. Real-valuedness of α near z_0 implies

$$q(z) = (z - z_0)^k g_0(z)$$

with g_0 smooth, which implies the smoothness of α at z_0 . So α can be smoothly extended to the whole Σ . \square

Corollary 4.2. *Umbilic points of a Lagrangian Bonnet pair are isolated. The umbilic set coincides with the zero set of h , i.e., $\mathcal{U} = \mathcal{U}_h$.*

The number k which is defined above is called the *index* of the umbilic point. We call the zero divisor $D = (h)$ of h the *Umbilic divisor* of a Lagrangian Bonnet pair.

In exactly the same way as in the case of Bonnet pairs in \mathbb{R}^3 and $\mathbb{C}P^2$ (see [1], [15]), for compact Riemann surfaces, Propositions 3.1, 4.1 imply the following

- Proposition 4.3.** (1) *There are no Lagrangian Bonnet pairs of genus zero.*
 (2) *Lagrangian Bonnet pairs of genus one have no umbilic points.*
 (3) *If Lagrangian Bonnet pairs of genus $g \geq 1$ exist, the umbilic divisor D is of degree $6g - 6$ and its class is $D = 3K$, where K is the canonical divisor.*

Acknowledgments. The first author was supported by NSFC grant No. 10701007 and No. 11071018. The second author was supported by NSFC grant No. No. 11271213. And the third author was supported by the NSFC (Grant Nos. 10941002, 11001262) and the Starting Fund for Distinguished Young Scholars of Wuhan Institute of Physics and Mathematics (Grant No. O9S6031001). The third author would like to thank the Hong Kong University of Science & Technology for the support during the project.

REFERENCES

1. A.I. Bobenko, *Exploring surfaces through methods from the theory of integrable systems. Lectures on the Bonnet problem*. SFB288 Preprint 403 (1999), math.DG/9909003.
2. O. Bonnet, *Mémoire sur la théorie des surfaces applicables*, J. Éc. Polyt. **42** (1867), 72-92.
3. I. Castro, C. R. Monteleagre and F. Urbano, *Closed conformal vector fields and Lagrangian submanifolds in complex space forms*, Pacific J. Math., **199** (2001), 269-301.
4. I. Carstro and F. Urbano, *Lagrangian surfaces in the complex Euclidean plane with conformal Maslov form*, Tôhoku Math. J. **45** (1993), 565-582.
5. I. Carstro and F. Urbano, *Twistor holomorphic lagrangian surfaces in the complex projective plane and hyperbolic planes*, Ann. of Glob. Anal. and Geom. **13** (1995), 59-67.
6. I. Carstro and F. Urbano, *On twistor harmonic surfaces in the complex projective plane*, Math. Proc. Camb. Phil. Soc., **122**, 1997, 115-129.
7. E. Cartan, *Sur les couples de surfaces applicables avec conservation des courbures principales*, Bull. Sci. Math. II. Ser. **66** (1942), 55-72 and 74-85.
8. S.S. Chern, *Deformation of surfaces preserving principal curvatures*, In I. Chavel and H.M. Farkas, editors, Differential Geometry and Complex Analysis, pages 155-163. Springer, New York, 1985.
9. W. H. Chen and H. Z. Li, *Bonnet surfaces and isothermic surfaces*, Result Math. (31)1997, 40-52.
10. X. Chen and C. K. Peng, *Deformations of surfaces preserving principal curvatures*, Lecture Notes in Math., Vol(1369), Springer Verlag, 1989, 63-70.
11. S.S. Chern and J.G. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math. **105** (1983), 59-83.
12. P. Dazord, *Sur la géométrie des sous-fibrés et des feuilletages lagrangiense*, Ann Sci. Éc. Norm. Super., **13** (1981), 465-480.

13. J. Eells and J.C. Wood, *Harmonic maps from surfaces to complex projective spaces*, Adv. in Math. **49** (1983), 217-263.
14. J.A. Gálvez, A. Martínez and P. Mira, *The Bonnet problem for surfaces in homogeneous 3-manifolds*, math.DG/0612766.
15. H.X. He and H. Ma, *Lagrangian Bonnet pairs in \mathbb{CP}^2* , Proc. Amer. Math. Soc. **137** (2009), 2725-2731.
16. F. Hélein F and P. Romon, *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces*, Differential Geometry and Integrable Systems, Contemp. Math. ,(Guest, Miyaoka and Ohnita ,Editors) Amer. Math. Soc.,Providence, (2002), 161-178.
17. G. Kamberov, F. Pedit and U. Pinkall, *Bonnet pairs and isothemic surfaces*, Duke Math. J. **92**(3), (1998)637-644.
18. H.B. Lawson and R. Tribuzy, *On the mean curvature function for compact surfaces*, J. Diff. Geom. **16** (1981) 179-183.
19. H. Ma , *Hamiltonian stationary Lagrangian surfaces in \mathbb{CP}^2* , Ann. Global Anal. Geom., **27** (2005), 1-16.
20. H. Ma and Y. Ma, *Totally real minimal Lagrangian tori in \mathbb{CP}^2* , Math. Z. **249** (2005), 241-267.
21. A. Ros and F. Urbano, *Lagrangian submanifolds of \mathbb{C}^n with conformal Maslov form and the Whitney sphere*, Invent. Math.,**101**, 1990, 501-519.
22. Y.G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), 501-519.
23. R. Schoen and J. Wolfson, *Minimizing volume among Lagrangian submanifolds*, in ‘Differential Equations: La Pietra 1996’, (ed.: Giaquinta, Shatah and Varadhan), Proc. Symp. Pure Math **65** (1999), 181-199.
24. B.A. Springborn, *Bonnet pairs in the 3-sphere*, Differential geometry and integrable systems (Tokyo, 2000), 297-303, Contemp. Math., 308, Amer. Math. Soc., Providence, RI, 2002.
25. R. Tribuzy, *A characterization of tori with constant mean curvature in space form*, Bol. Soc. Bras. Math. **11** (1980), 259-274.

SCHOOL OF MATHEMATICS AND SYSTEMS SCIENCE OF BEIJING UNIVERSITY OF AERONAUTICS AND ASTRONAUTICS
, LIMB OF THE MINISTRY OF EDUCATION, BEIJING 100083, P.R. CHINA

E-mail address: hehx@buaa.edu.cn

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, 100084, P.R. CHINA

E-mail address: hma@math.tsinghua.edu.cn

DEPARTMENT OF MATHEMATICS, HONG KONG UNIV. OF SCI. AND TECH., CLEAR WATER BAY, KOWLOON,
HONG KONG

E-mail address: maexwang@ust.hk